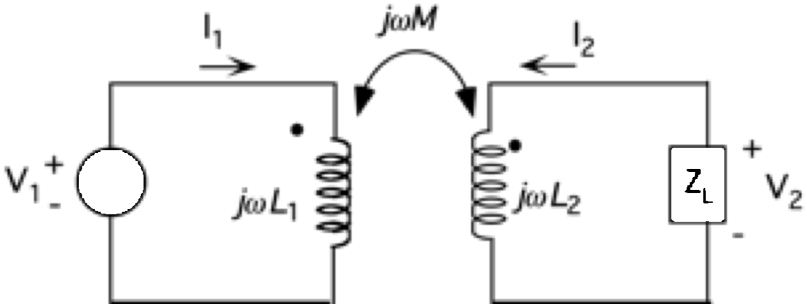
The Ideal Transformer

For the coupled inductor circuit shown below:



From KVL around both loops, we obtain:

$V_1 = j\omega L_1 I_1 + j\omega M I_2$ and $V_2 = j\omega M I_1 + j\omega L_2 I_2$

where $M = k\sqrt{L_1 L_2}$ From Ohm's law, we also have $V_2 = -Z_L I_2$. Substituting and rearranging, we find the following relationships between the voltages and currents:

$$\frac{V_1}{V_2} = \frac{1}{k}\sqrt{\frac{L_1}{L_2}} + \frac{j\omega}{Z_L}\frac{\left(1-k^2\right)}{k}\sqrt{L_1 L_2}$$

$$\frac{I_1}{I_2} = \frac{-j\omega L_2 - Z_L}{j\omega k\sqrt{L_1 L_2}}$$

Interestingly as $k \to 1$, the voltage expression becomes:

$$\frac{V_1}{V_2} = \sqrt{\frac{L_1}{L_2}} = \frac{N_1}{N_2} \qquad \text{when } k \to 1$$

where N_1 and N_2 are the number of turns in each inductor. Thus, when the inductors are maximally coupled, the ratio of the winding voltages equals the turns ratio, independent of the load impedance Z_L.

Also, if $k \to 1$, the current expression becomes:

$$\frac{I_1}{I_2} = \frac{-j\omega L_2 - Z_L}{j\omega\sqrt{L_1 L_2}}$$

Further, if $\left|j\omega L_2\right| >> Z_L$, the current expression becomes:

$$\frac{I_1}{I_2} = -\sqrt{\frac{L_2}{L_1}} = -\frac{N_2}{N_1} \quad \text{when } |j\omega L_2| >> Z_L \text{ and } k \to 1$$

In order for coupled inductors to be considered an ideal transformer, both conditions must apply: $|j\omega L_2| >> Z_L$ and $k \to 1$, which means that the self-inductances of the windings have to be large, and the same magnetic flux must pass through both inductors.

When both conditions are satisfied, the net energy stored by the transformer is zero, indicating a net inductance of zero. We can see this from:

$$W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + k\sqrt{L_1 L_2} I_1 I_2$$

but, when $\frac{I_1}{I_2} = -\sqrt{\frac{L_2}{L_1}} = -\frac{N_2}{N_1}$, this becomes

$$W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 \left(-\sqrt{\frac{L_1}{L_2}}\right)^2 I_1^2 - \sqrt{L_1 L_2}\sqrt{\frac{L_1}{L_2}} I_1^2 \to 0$$

Finally, it is possible to design a transformer in which the winding voltages follow the ideal transformer ratio, but the currents do not. This occurs when k=1 but the load impedance Z_L is not negligible to the secondary inductive reactance. This is often accomplished by adding an air gap to the core. These transformers are used in flyback power supplies, where the "flyback" is a voltage produced by the transformer when the current at the input is switched suddenly.